Abstract Algebra
Solution of Assignment-1

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[M.Sc. Tech Mathematics]

1. Illustrate Cayley's Theorem by calculating the left regular representation for the group $V_4 = \{e, a, b, c\}$ where $a^2 = b^2 = c^2 = e, ab = ba = c, ac = ca = b, bc = cb = a$.

   Solution :
   Let $V_4 = \{e, a, b, c\}$. Now computing the permutation σ_g induced by the action of left-multiplication by the group element a.
   a.e = ae = a and so $\sigma_g(e)$ = a
   a.a = aa = a^2 = e and so $\sigma_g(a)$ = e
   a.b = ab = c and so $\sigma_g(b)$ = c
   a.c = ac = b and so $\sigma_g(c)$ = b
   Hence σ_a = (ea)(bc).

   Now computing σ_g induced by the action of left-multiplication by the group element b.
   b.e = be = b and so $\sigma_g(e)$ = b
   b.a = ba = c and so $\sigma_g(a)$ = c
   b.b = bb = b^2 = e and so $\sigma_g(b)$ = e
   b.c = bc = a and so $\sigma_g(c)$ = a
   Hence σ_b = (eb)(ac).

   Similarlly Computing σ_g induced by the action of left-multiplication by the group element c.
   c.e = ce = c and so $\sigma_g(e)$ = c
   c.a = ca = b and so $\sigma_g(a)$ = b
   c.b = cb = a and so $\sigma_g(b)$ = a
   c.c = cc = c^2 = e and so $\sigma_g(c)$ = e
   Hence σ_c = (ec)(ab).

   Which explicitly gives the permutation representation $V_4 \to V_4$ associated to this action.

2. Show that A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.

Solution :

Since we can decompose any permutation into a product of disjoint cycle. In S_5 , since disjoint cycle commutes. Let $V_5 = \{e, a, b, c, d\}$ Here an element of S_5 must have one the following forms:

(i) $(abcde)$ - even

(ii) $(abc)(de)$ - odd (even P * odd P)

(iii) (abc) - even

(iv) $(ab)(cd)$ - even (odd P * odd P)

(v) (ab) - odd

(vi) (e) -even

So element of A_5 is of the form (i), (iii), (iv) and (vi). As we know that, when a permutation is written as disjoint cycles, it's order is the lcm (least common multiple) of the lengths of the cycles.

(i) $(abcde)$ has order 5

(iii) (abc) has order 3

(iv) $(ab)(cd)$ has order 2

(vi) (e) has order 1

Now since elements of order 5 in A_5 are of the form (i). There are 5! distinct expression for cycle of the form $(abcde)$ where all a, b, c, d, e are distinct. since expression representation of the element of type $(abcde) = (bcdea) = (cdeab) = (deabc) = (eabcd)$ are equivalent. So total elements of order 5 are $\dfrac{5 \times 4 \times 3 \times 2 \times 1}{5} = 24$.

Now for elements of order 3. Since elements of order 3 in A_5 is of the form (abc). Here there are 5 choices for a, 4 choices for b and 3 choices for c. so there are $5 \times 4 \times 3 = 60$ possible ways to write such a cycle. Since expression representation of the element of type $(abc) = (bca) = (cab)$ are equivalent.So total no. of elements of order 3 in A_5 are $\dfrac{60}{3} = 20$.

Here since even permutation of order 2 are of the form $(ab)(cd)$. so there are $5 \times 4 \times 3 \times 2$ ways to write such permutation. Since disjoint cycles commute there, so there are 8 different ways that differently represent the same permutations :-

$(ab)(cd) = (ab)(dc) = (ba)(dc) = (ba)(cd) = (cd)(ab) = (dc)(ab) = (dc)(ba) = (cd)(ba)$.

So there are $\dfrac{5 \times 4 \times 3 \times 2}{8} = 15$ elements of order 2.

{No. of ways of selecting r different things out of n is nPr }

3. Show that if $n \geq m$ then the number of m-cycles in S_n is given by $\frac{n(n-1)(n-2)...(n-m+1)}{m}$.

Proof :
For any given S_n, there are n elements in $S_n = \{1, 2, 3, ...m...n\}$. so we must have n-choices for 1st element, then n-1 choices for 2nd element, n-2 choices for 3rd element and so on... and we have n-m+1 choices for m^{th} element etc. So there are total no. of n(n-1)(n-2)...(n-m+1) for a m-cycles.
Now we want to count m-cycles in S_n, since for 2-cycles $(ab) = (ba)$
{two equivalent notation , i.e same permutation}
For 3-cycles $(a, b, c) = (b, c, a) = (c, a, b)$ {i.e 3-equivalent notation}
For 4-cycles $(a, b, c, d) = (b, c, d, a) = (c, d, a, b) = (d, a, b, c)$ {four equivalent notation}
- - - - - - - - - - -
Similarly for m-cycles there are m-equivalent notation for any permutations.
Now, Since we have, $n(n-1)(n-2)...(n-m+1)$ choices to form a m-cycle in which there are m-equivalent notations for any permutation of length m.
So the no. of m-cycles in S_n is

$$\frac{n(n-1)(n-2)...(n-m+1)}{m}$$

4. Let σ be the m-cycle $(12 \dots m)$. Show that σ^i is also an m-cycle if and only if i is relatively prime to m.

Proof :
First we note that if τ is k cycle then $|\tau| = k$
since $\sigma^i(x) \equiv$ x+i mod m for any x, $1 \leq x \leq m$

Claim : $\sigma^i = (\sigma^i(1)\sigma^i(2)...\sigma^i(m))$

<u>we prove it by contradiction</u>

Let i=1. Then the statement is obviously true.
Suppose that

$$\sigma^{i-1} = (\sigma^{i-1}(1)\sigma^{i-1}(2)...\sigma^{i-1}(m))$$

then $\sigma^i = \sigma(\sigma^{i-1}) = \sigma\{\sigma^{i-1}(1)...\sigma^{i-1}(m)\}$

Since, here σ sends $\sigma^{i-1}(i)$ to $\sigma^i(1)$,
thus $\sigma^i = (\sigma^{i-1}(1)...\sigma^i(m))$
$\implies \sigma^i = (\sigma^{i-1}(1)...\sigma^i(m))$
Since $\sigma^i(m) \equiv$ m+i mod m $\equiv$ i mod m and $\sigma^{i-1}(1) \equiv$ 1+i-1 mod m $\equiv$ i mod m

i.e $\sigma^i(m) = \sigma^{i-1}(1)$
$\implies \sigma^i$ is an m-cycle.

Converse part
Suppose σ^i is an m-cycle and suppose that $(i, m) = d > 1$. (we prove it by contradiction)
then there exists k,n $\in \mathbb{N}$ such that i=kd and m=nd,
since, $(\sigma^i)^n = (\sigma^{kd})^n = \sigma^{kdn} = \sigma^{mk} = (\sigma^m)^k = I$
where I is the identity permutation.
Hence $|\sigma^i| \leq n < m$.
which is contradiction, since σ^i is an m-cycle and thus $|\sigma^i| = m$. Thus i is relatively prime to m.

5. **Que. No.05** Let $n \geq 3$. Prove the following in S_n.
   (a) Every permutation of S_n can be written as a product of at most $n - 1$ transpositions.
   (b) Every permutation of S_n that is not a cycle can be written as a product of at most $n - 2$ transpositions.

Proof (a) :

We know that if $k \geq 2$, the cycle $(a_1, a_2, ...a_k)$ can be written as $(a_1, a_k)(a_1, a_{k-1})...(a_1, a_2)$ which is k-1 transpositions.
Case-I, If k=1, then this cycle is the trivial cycle or the identity, which can be written as 1-1=0 transpositions
Case-II, if $k > 1$,
we know that every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles, thus we can write
$\sigma = (a_{11}, a_{12}, ..., a_{1k_1})(a_{21}, a_{22}, ..., a_{2k_2})...(a_{m1}, a_{m2}, ..., a_{mk_m})$
where $k_1 + k_2 + ... + k_m = n$ and each of these cycle is disjoint.
we know that cycle i can be written as a product of $k_i - 1$ transpositions, and $\sum_{i=1}^{m}(k_i - 1) = \sum_{i=1}^{m} k_i - \sum_{i=1}^{m} 1 = n - m$, this is maximized when m is minimized and the least value of m is 1.
Thus, the largest value of n-m can be n-1.

Proof (b) :

From part (a), $\sigma = (a_{11}, a_{12}, ..., a_{1k_1})(a_{21}, a_{22}, ..., a_{2k_2})...(a_{m1}, a_{m2}, ..., a_{mk_m})$ where $\sum_{i=1}^{m} k_i = n$ and each of cycles is disjoint and also from (a), we still know that cycles i can be written as a product of $k_i - 1$ transpositions and
$\sum_{i=1}^{m}(k_i - 1) = \sum_{i=1}^{m} k_i - \sum_{i=1}^{m} 1 = n - m$, However, since σ is not a cycle. $m \geq 2$, thus n-m is maximized when m is minimized i.e m=2 i.e n-2 is the maximum value of n-m.
Hence every permutation of S_n that is not a cycle can be written as a product of at most n-2 transpositions.

6. **Que. No.06** Let σ be a permutation of a set A. We say that σ moves $a \in A$ if $\sigma(a) \neq a$. Let S_A denote the permutations on A.

   (a) If A is a finite set then how many elements are moved by a n-cycle $\sigma \in S_A$?

   (b) Let A be an infinite set and let H be the subset of S_A consisting of all $\sigma \in S_A$ such that σ only moves finitely many elements of A. Show that $H \leq S_A$.

   (c) Let A be an infinite set and let K be the subset of S_A consisting of all $\sigma \in S_A$ such that σ moves at most 50 elements of A. Is $K \leq S_A$? Why?

   Proof (a):
   If A is finite, then σ moves only n elements because σ is n-cycle and the elements which is not in cycle are fixed.
   Proof (b):
   We may prove it by One-Step Subgroup Test.
   As A is infinite set and $\sigma \in S_A$ moves only finitely many elements of A. Since H consists all $\sigma \in S_A$
   $\Rightarrow$ H is non-empty.
   Now let, $\sigma \in H \implies \sigma^{-1} \in H$.
   So, $\sigma o \sigma^{-1} = I = \in H$
   Now checking for closure property,
   Let σ_1 and $\sigma_2 \in H$ be any two permutations such that σ_1 and σ_2 both moves only finitely many elements of A.
   Then $\sigma_1 \text{o} \sigma_2$ also moves only finitely many elements of A.
   $\Rightarrow$ Closure property holds.
   $\Rightarrow$ H is subgroup of A_5.

   Proof (c):
   No, K will not be subgroup of S_A
   Because, suppose that σ_1 moves at most 50 elements and σ_2 moves at most 50 elements, then $\sigma_1 o \sigma_2$ (Product of two permutations) might moves more than 50 elements.
   $\Rightarrow$ Closure property with respect to function composition is not satisfied in K.
   $\Rightarrow$ K is not a subgroup of S_A.

7. **Que. No.07** Show that if σ is a cycle of odd length then σ^2 is a cycle.

   Proof : Suppose $\sigma : A \to A$ is a cycle with odd length. Then we can write σ in a cycle notation as σ

   $\sigma = (a_1, a_2, ..., a_{ak+1})$ where $a_1, a_2, ..., a_{2k+1} \in A$
   On simple calculation, we may show that

   $\sigma^2 = (a_1, a_2, ...a_{2k+1})(a_1, a_2, ...a_{2k+1})$
   $\sigma^2 = (a_1, a_3, a_5, ...a_{2k+1}, a_2, a_4...a_{2k})$
   $\implies \sigma^2$ is cycle whenever σ is cycle.

8. **Que. No.08** Let p be a prime. Show that an element has order p in S_n if and only if its cycle decomposition is a product of commuting p-cycles. Show by an explicit example that this need not be the case if p is not prime.

Proof :

⇒ Suppose the order of σ is p(p is prime).
Since order of σ is the lcm of the sizes of the disjoint cycles in the cycle decomposition of σ, So all of these cycle must have sizes that divides p is either 1 or p.

Since 1-cycles are omitted from the notation for the cycle decomposition of σ. Thus the cycle decomposition consists entirely of p-cycles. Thus σ is the product of disjoint commuting p-cycles.

⇐ Suppose σ is the product of disjoint p-cycles. i.e $\sigma = c_1c_2c_3...c_r$

then $\sigma^p = (c_1c_2c_3...c_r)^2 = c_1^pc_2^pc_3^p...c_r^p = 1$

(since the p^{th} power of p-cycles in σ are all 1, so their product is 1)

$\sigma^p = 1$

A p-cycle has order p, so no smaller power of σ can be 1. Hence $|\sigma| = p$.

For an example :
Showing these conclusions may fail when p is not a prime.

Let p=6, $\sigma = (12)(345)$
$|\sigma| = lcm(2,3) = 6$
but σ is not the product of commuting 6-cycles.

9. **Que. No.09** Show that if $n \geq 4$ then the number of permutations in S_n which are the product of two disjoint 2-cycles is $n(n-1)(n-2)(n-3)/8$.

Solution :
Given n$\geq$ 4.
Since, Permutations which are the product of two disjoint 2-cycles is of the form (ab)(cd), i.e of length 4.
Hence, there are n choices for a, (n-1) choices for b, (n-2) choices for c and (n-3) choices for d.
So there are $n(n-1)(n-2)(n-3)$ possible ways to write to write such a cycle.
Since disjoint cycles commutes there, so there are 8 different ways that differently represent the same cycle(As i mentioned it in sol. of Que.2)
Hence total number of Permutation in S_n which are the product of two disjoint 2-cyles is $\dfrac{(n)(n-1)(n-2)(n-3)}{8}$.

10. **Que. No.10** Let $b \in S_7$ and suppose $b^4 = (2143567)$. Find b.

Solution :

$$\because b \in S_7$$
$$|b| = 7$$
$$\Rightarrow b^7 = I$$
$$\text{So } b = Ib = (b^7).b = b^8 = (b^4)^2$$
$$\Rightarrow b = b^4.b^4$$
$$\Rightarrow b = (2143567)(2143567)$$
$$= (2457136).$$

As given that $b^4 = (2143567)$.

11. **Que. No.11** Let $b = (123)(145)$. Write b^{99} in disjoint cycle form.

Solution :

Since $b = (123)(145) = (14523)$. So order of b is 5.
(In case of single cycle. The order of permutation is the degree of permutation is the lengths of the set.)
Now since $|b| = 5$, then $b^5 = I$.
So we can write $b^{99} = (b^5)^{19}.b^4 = Ib^4 = b^4 = b^{-1}$.
Since $b = (14523) \Rightarrow b^4 = b^{-1} = (32541) = (132541)$
so $b^{99} = (13254)$ or $(154)(132)$.

12. **Que. No.12** Find three elements σ in S_9 with the property that $\sigma^3 = (157)(283)(469)$.

Solution :

Let $1 = a_1$, $2 = a_2$, $3 = a_3$, $4 = a_4$, $5 = a_5$, $6 = a_6$, $7 = a_7$ and $8 = a_8$.
Now we have to find σ such that $\sigma^3 = (a_1a_5a_7)(a_2a_8a_3)(a_4a_6a_9)$
then $\sigma_1 = (a_1 \text{ .... } a_5 \text{ .... } a_7 \text{ .... })$
$\sigma_1 = (a_1\ a_2 \text{ .. } a_5\ a_8 \text{ .. } a_7\ a_3 \text{ .. })$
$\sigma_1 = (a_1\ a_2\ a_4\ a_5\ a_8\ a_6\ a_7\ a_3\ a_9)$
$\sigma_1 = (1\ 2\ 4\ 5\ 8\ 6\ 7\ 3\ 9)$.
Similarly we can find other two elements
$\sigma_2 = (a_1 \text{ .... } a_5 \text{ .... } a_7 \text{ .... })$
$\sigma_2 = (a_1\ a_3 \text{ .. } a_5\ a_2 \text{ .. } a_7\ a_8 \text{ .. })$
$\sigma_2 = (a_1\ a_3\ a_9\ a_5\ a_2\ a_4\ a_7\ a_8\ a_6)$
$\sigma_2 = (1\ 3\ 9\ 5\ 2\ 4\ 7\ 8\ 6)$.
and
$\sigma_3 = (a_2 \text{ .... } a_8 \text{ .... } a_3 \text{ .... })$
$\sigma_3 = (a_2\ a_1\ a_4\ a_8\ a_5\ a_6\ a_3\ a_7\ a_9)$
$\sigma_3 = (2\ 1\ 4\ 8\ 5\ 6\ 3\ 7\ 9)$.

13. **Que. No.13** Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of the members are even.

Solution :

Let H $\subset S_n$ be any subgroup.
Now, we define $\overline{H} = \{\sigma \in$ H — σ is even $\}$

Claim: $\overline{H}$ is subgroup of H.

Let f,g $\in \overline{H}$, Since g are even, so g^{-1} is also even.
since the product of even permutations are still even, so we have fog^{-1} is even.
So, here there are only two possibilities either $\overline{H} = H$ or $\overline{H} \subsetneq H$

Case-I, if $\overline{H} = H$, then we are done.

Case-II, if $\overline{H} \neq H$, then we need to show that $|\overline{H}| = \dfrac{|H|}{2}$
Since $\overline{H} \neq H$,it implies that there exists at least one odd permutations $\sigma \in H$
Now consider f: $\overline{H} \to \dfrac{H}{\overline{H}}$ defined by f(h) = $\sigma.h$ for any h $\in \overline{H}$.
since σ is odd and h is even
$\Rightarrow \sigma.h$ is odd.
$\Rightarrow \sigma.h \in \dfrac{H}{\overline{H}}$
To prove that $\overline{H} = \frac{|H|}{2}$, We need to prove f is 1-1 and onto.
for 1-1
let $h_1, h_2 \in H$ such that $h_1 = h2$.
since $h_1 = h_2$
$\Rightarrow \sigma h_1 = \sigma h_2 \Rightarrow f(h_1) = f(h_2) \Rightarrow$ f is 1-1.
and for onto
since $f^{-1} : \frac{H}{\overline{H}} \to \overline{H}$ is given by $f^{-1}(h) = \sigma^{-1}h'$ for every h' $\in \dfrac{H}{\overline{H}}$.
So f is both 1-1 and onto
$\Rightarrow |\overline{H}| = |\dfrac{H}{\overline{H}}|$, hence $|\overline{H}| = \dfrac{|H|}{2}$

14. **Que. No.14** Suppose that H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n. rate S_n.

Proof :

Let H be a subgroup of S_n of odd order.
i.e $|H|$ = odd order
We may prove it by contradiction.
To the contrary, suppose H $\nsubseteq A_n$, then
suppose $\exists\ \sigma \in$ H such that σ is an odd permutation.
Let H = $\{\underbrace{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p}_{\text{Odd}}\} \cup \{\underbrace{\beta_1, \beta_2, \beta_3, ..., \beta_q}_{\text{Even}}\}$

$\therefore \sigma$H = $\{\underbrace{\sigma\alpha_1, \sigma\alpha_2, \sigma\alpha_3, ..., \sigma\alpha_p}_{\text{Even}}\} \cup \{\underbrace{\sigma\beta_1, \sigma\beta_2, \sigma\beta_3, ..., \sigma\beta_q}_{\text{Odd}}\}$

= H
$\implies p = q$
$\implies |H|$ = 2p = 2q = even
Which is a contradiction.
$\implies H \subset A_n$

15. **Que. No.15** Prove that the smallest subgroup of S_n containing (12) and $(12 \ldots n)$ is S_n. In other words, these generate S_n.

Proof :

Let $\sigma = (12)$ and $\tau = (123...n)$
Suppose H is subgroup of S_n which contains both $\sigma = (12)$ and $\tau = (123...n)$.
Now, we need to show that H = S_n.
Clearly, we have $H \subseteq S_n$. Since subgroups in particular are subsets.
Since we know that S_n is generated by (n-1) transpositions (12)(23)(34)(45)...(n-1 n).
Now, I want to show that (12) and (123...n) generates these (n-1) transposition.
Consider, $\tau\sigma\tau^{-1}$
$(12...n)(12)(12...n)^{-1} = (23)$
$(12...n)(23)(12...n)^{-1} = (34)$
$(12...n)(34)(12...n)^{-1} = (45)$
. . . .
$(12...n)(n-2 \quad n-1)(12...n)^{-1} = (n-1 \quad n)$
$(12...n)(n-1 \quad n)(12...n)^{-1} = (n \quad 1)$

Now i prove it by induction...
for n = 1, it is obviously true.
We assume that it is true for n = k, then
$(12...k)(k-1 \quad k)(12...k)^{-1} = (k \quad 1)$
Now, we wish to show that it is true for n = k+1
$(1, 2, ..., k, k+1)(k, k+1)(1, 2, ..., k, k+1)^{-1}$
$= (1, 2, ..., k, k+1)(k+1, k)(k+1, k, ..., 3, 2, 1)$
$= 6(1, 2, ..., k, k+1)(k+1)(k, ..., 3, 2, 1)$
$= (1, 2, ..., k, k+1)(k, ..., 3, 2, 1)$
$= (k)(k-1)...(3)(2)(1)(1, k+1)$
= (k+1, 1)
So, it is true for n=k+1
$\Rightarrow$ (12) and (123...n) generates S_n
Which shows that $S_n \subseteq H$.
Thus h = S_n

16. **Que. No.16** Prove that for $n \geq 3$ the subgroup generated by the 3-cycles is A_n.

Proof :

Since every 3-cycle is an even permutation, then every 3-cycle of S_n is in A_n.

Now, Let $\tau \in A_n \Rightarrow \tau$ is an even permutation.
$\Rightarrow \tau$ is a product of an even no. of transposition.
However, $(a_1a_2)(a_3a_4) = (a_1a_2a_3)(a_2a_3a_4)$
And $(a_1a_2)(a_1a_3) = (a_1a_3a_2)$
Consequently, every product of two transposition(whether they share an element or not) can be written as a product of 3-cycles.
Hence, τ can be written as a product of 3-cycles.
$\Rightarrow$ For any n$\geq$ 3, the subgroup generated by 3-cycle is A_n.

17. **Que. No.17** Prove that if a normal subgroup of A_n contains even a single 3-cycle it must be all of A_n.

Proof :
Let N $\subset A_n$ be Normal subgroup and suppose that (abc) $\in$ N. Let $\sigma' \in A_n$ be an arbitrary 3-cycles.
Then $\sigma' = \tau(abc)\tau^{-1}$ for some $\tau \in S_n$.
Now here, there are two possibility either $\tau \in A_n$ or $\tau \notin A_n$.
Case -I, If $\tau \in A_n$ then $\sigma' \in$ N and we are done.
Case -II, If $\tau \notin A_n$ then $\tau' = \tau(ab)$ is in A_n and $\tau' = \tau(acb)\tau'^{-1}$ is once again in N.
$\Rightarrow$ If N $\trianglelefteq A_n$ and contains a 3-cycle. Then N=A_n.

18. **Que. No.18** Prove that A_5 has no non-trivial proper normal subgroups. In other words show that A_5 is a simple group.

Solution :
Order of $A_5 = |A_5| = \dfrac{5!}{2} = 60 = 2^2.3.5.$
Let N be proper normal subgroup of A_5, then
$|N| = 2 , 3 , 4 , 5 , 6 , 10 , 12 , 15 , 20 , 30.$
Total no. of 5 order elements in $A_5 = \dfrac{5P5}{5} = 24,$
Total no. of elements of 3 order in $A_5 = \dfrac{5P3}{5} = 20,$
And total no. of 15-order elements in $A_5 = 0$.
Let us assume that $|H| = 3 , 6 , 12 , 15$
then $|\dfrac{A_5}{H}| = 20 , 10 , 5 , 4$
so $\gcd\left(3 , |\dfrac{A_5}{H}|\right) = 1$
$\Longrightarrow$ H would contain all 20 elements of order 3.
Which is a contradiction.
{ As, Theorem says that If H be Normal subgroup of a finite group G. And if $\gcd\left(|x|, |\dfrac{G}{H}|\right)$=1, then $x \in$ G}.

Similarly, suppose that $|H| = 5$,10 , 20
then $|\frac{A_5}{H}| = 12$, 6 , 3
$\implies$ H would contain all 24 elements of order 5.
which is a contradiction.
Let $|H| = 30$, then $|\frac{A_5}{H}| = 2$.
So again $\gcd\left(3\,, |\frac{A_5}{H}|\right) = 1$ and $\gcd\left(5\,, |\frac{A_5}{H}|\right) = 1$.
$\implies$ H would contain all 20+24 = 44 elements.
we get again a contradiction.
And finally, let us assume that, $|H|$= 2 or 4.
$\implies |\frac{A_5}{H}| = 30,\ 15$
Since, we know that any group of order 30 or 15 has an element of order 15.
or As, if $|\frac{A_5}{H}| = 15 = 3 \times 5 = p \times q$ where p=3 and q=5.
(Theorem : If G is a group of order pq, where p and q are primes, $p < q$ and p $\nmid$ q, then G is cyclic.)
$\Rightarrow$ G has at least one element of order 15.
Which is again contradiction,
because A_5 contains no such element, neither does $\frac{A_5}{H}$.
This proves that A_5 is simple.

19. **Que. No.19** Show that $Z(S_n)$ is trivial for $n \geq 3$.

Solution :

Let $\sigma \in S_n$ be a non-identity element then there exists two distinct a,b $\in$ $\{1, 2, 3, ..., n\}$ with $\sigma(a) = b$.
Since n $\geq$ 3, Now choosing k $\in \{1, 2, 3, ..., n\}$ such that k $\neq$ a and k $\neq$ b.
Let $\tau = (ak)$. Then
$\tau(\sigma(a)) = \tau(b) = k$ and $\sigma(\tau(a)) = \sigma(a) = b$
since k$\neq$b $\Rightarrow \tau(\sigma(a)) \neq \sigma(\tau(a))$.
Hence for every non-identity permutation in S_n, there exists some element not commuting with it.
Therefore $Z(S_n)$ must be trivial.

20. **Que. No. 20** Show that two permutations in S_n are conjugate if and only if they have the same cycle structure or decomposition. Given the permutation $x = (12)(34)$, $y = (56)(13)$, find a permutation a such that $a^{-1}xa = y$.

Proof :

For any σ and any d $\leq$ n, we have
$\sigma(12...d)\sigma^{-1} = (\sigma(1)\sigma(2)....\sigma(d))$

This shows that any conjugate of d-cycle is again d-cycle.
Since every permutation is a product of disjoint cycles, it follows that the cycle structure of conjugate permutations are the same.
In other direction,

Let $\tau = (a_1a_2.....a_r)(a_{r+1}a_{r+2}.....a_s)....(a_l.....a_m)$ and
$\tau' = (a'_1a'_2.....a_r)(a_{r+1}a_{r+2}.....a_s)....(a_l.....a_m)$
be two permutations having the same cycle structure.
Define $\sigma \in S_n$ by $\sigma(a'_i) = a'$ for i = 1,2,...,m then
$\sigma\tau\sigma^{-1} = \sigma(a_1a_2....a_r)\sigma^{-1}\sigma(a_{r+1}a_{r+2}....a_s)\sigma^{-1}....\sigma(a_l....a_m)\sigma^{-1}$
$= (a'_1a'_2.....a_r)(a_{r+1}a_{r+2}.....a_s)....(a_l.....a_m)$
$= \tau'$
This shows that τ and τ' are conjugate.
Now, Given the permutation $x = (12)(34)$, $y = (56)(13)$
Since that $a^{-1}xa = y$.
$\therefore xa = ay \Rightarrow x = aya^{-1}$.
$\Rightarrow ((12)(34)) = a((56)(13))a^{-1}$
$\Rightarrow ((12)(34))(5)(6) = a((56)(13)(2)(4))a^{-1}$
. $= (a(5)a(6))(a(1)a(3))a(2)a(4)$
$\Rightarrow 1 = a(5), 2 = a(6), 3 = a(1), 4 = a(3)$ and $5 = a(2), 6 = a(4)$

$$\Rightarrow a = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 6 & 1 & 2 \end{matrix}$$

$\Rightarrow a = (134625)$
Checking for a, $a = (134625)$ and $a^{-1} = (526431) = (152643)$
$\therefore a^{-1}xa = (134625)((12)(34))(152643)$
$= (13)(2)(4)(56) = (13)(56) =$ RHS, Hence done.

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