Abstract Algebra
Solution of Assignment-1

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[ M.Sc. Tech Mathematics ]

1. Illustrate Cayley’s Theorem by calculating the left regular representation for the group $V_4 = \{e, a, b, c\}$ where $a^2 = b^2 = c^2 = e, ab = ba = c, ac = ca = b, bc = cb = a.$

**Solution:**
Let $V_4 = \{e, a, b, c\}$. Now computing the permutation $\sigma_g$ induced by the action of left-multiplication by the group element $a$.

- $a.e = ae = a$ and so $\sigma_a(e) = a$
- $a.a = aa = a^2 = e$ and so $\sigma_a(a) = e$
- $a.b = ab = c$ and so $\sigma_a(b) = c$
- $a.c = ac = b$ and so $\sigma_a(c) = b$

Hence $\sigma_a = (ea)(bc)$.

Now computing $\sigma_g$ induced by the action of left-multiplication by the group element $b$.

- $b.e = be = b$ and so $\sigma_b(e) = b$
- $b.a = ba = c$ and so $\sigma_b(a) = c$
- $b.b = bb = b^2 = e$ and so $\sigma_b(b) = e$
- $b.c = bc = a$ and so $\sigma_b(c) = a$

Hence $\sigma_b = (eb)(ac)$.

Similarly Computing $\sigma_g$ induced by the action of left-multiplication by the group element $c$.

- $c.e = ce = c$ and so $\sigma_c(e) = c$
- $c.a = ca = b$ and so $\sigma_c(a) = b$
- $c.b = cb = a$ and so $\sigma_c(b) = a$
- $c.c = cc = c^2 = e$ and so $\sigma_c(c) = e$

Hence $\sigma_c = (ec)(ab)$.

Which explicitly gives the permutation representation $V_4 \rightarrow V_4$ associated to this action.
2. Show that $A_5$ has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.

Solution:
Since we can decompose any permutation into a product of disjoint cycle. In $S_5$, since disjoint cycle commutes. Let $V_5 = \{e, a, b, c, d\}$ Here an element of $S_5$ must have one the following forms:

(i) $(abcde)$ - even
(ii) $(abc)(de)$ - odd (even P * odd P)
(iii) $(abc)$ - even
(iv) $(ab)(cd)$ - even (odd P * odd P)
(v) $(ab)$ - odd
(vi) $(e)$ - even
So element of $A_5$ is of the form (i), (iii), (iv) and (vi). As we know that, when a permutation is written as disjoint cycles, it’s order is the lcm (least common multiple) of the lengths of the cycles.

(i) $(abcde)$ has order 5
(ii) $(abc)(de)$ has order 3
(iii) $(ab)(cd)$ has order 2
(vi) $(e)$ has order 1

Now since elements of order 5 in $A_5$ are of the form (i). There are 5! distinct expression for cycle of the form $(abcde)$ where all a, b, c, d, e are distinct. since expression representation of the element of type $(abcde) = (bcdea) = (cdeab) = (deabc) = (eabcd)$ are equivalent. So total elements of order 5 are $\frac{5!}{8} = 24$.

Now for elements of order 3. Since elements of order 3 in $A_5$ is of the form $(abc)$. Here there are 5 choices for a, 4 choices for b and 3 choices for c. so there are $5 \times 4 \times 3 = 60$ possible ways to write such a cycle. Since expression representation of the element of type $(abc) = (bca) = (cab)$ are equivalent.So total no. of elements of order 3 in $A_5$ are $\frac{60}{3} = 20$.

Here since even permutation of order 2 are of the form $(ab)(cd)$. so there are $5 \times 4 \times 3 \times 2$ ways to write such permutation. Since disjoint cycles commute there, so there are 8 different ways that differently represent the same permutations :-


So there are $\frac{5 \times 4 \times 3 \times 2}{8} = 15$ elements of order 2.

{No. of ways of selecting r different things out of n is $nP_r$ }
3. Show that if \( n \geq m \) then the number of \( m \)-cycles in \( S_n \) is given by \( \frac{n(n-1)(n-2)\ldots(n-m+1)}{m} \).

**Proof:**

For any given \( S_n \), there are \( n \) elements in \( S_n = \{1, 2, 3, \ldots m, \ldots n\} \). So we must have \( n \)-choices for 1st element, then \( n-1 \) choices for 2nd element, \( n-2 \) choices for 3rd element and so on... and we have \( n-m+1 \) choices for \( m^{th} \) element etc. So there are total no. of \( n(n-1)(n-2)\ldots(n-m+1) \) for a \( m \)-cycles.

Now we want to count \( m \)-cycles in \( S_n \), since for 2-cycles \( (ab) = (ba) \) \{two equivalent notation , i.e same permutation\} 
For 3-cycles \( (a,b,c) = (b,c,a) = (c,a,b) \) \{i.e 3-equivalent notation\} 
For 4-cycles \( (a,b,c,d) = (b,c,d,a) = (c,d,a,b) = (d,a,b,c) \) \{four equivalent notation\}

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Similarly for \( m \)-cycles there are \( m \)-equivalent notation for any permutations.

Now, Since we have, \( n(n-1)(n-2)\ldots(n-m+1) \) choices to form a \( m \)-cycle in which there are \( m \)-equivalent notations for any permutation of length \( m \).

So the no. of \( m \)-cycles in \( S_n \) is 

\[
\frac{n(n-1)(n-2)\ldots(n-m+1)}{m}
\]

4. Let \( \sigma \) be the \( m \)-cycle \( (12\ldots m) \). Show that \( \sigma^i \) is also an \( m \)-cycle if and only if \( i \) is relatively prime to \( m \).

**Proof:**

First we note that if \( \tau \) is \( k \) cycle then \( |\tau| = k \)

since \( \sigma^i(x) \equiv x+i \mod m \) for any \( x, 1 \leq x \leq m \)

Claim : \( \sigma^i = (\sigma^i(1)\sigma^i(2)\ldots\sigma^i(m)) \)

we prove it by contradiction

Let \( i=1 \). Then the statement is obviously true.

Suppose that

\[
\sigma^{i-1} = (\sigma^{i-1}(1)\sigma^{i-1}(2)\ldots\sigma^{i-1}(m))
\]

then \( \sigma^i = \sigma(\sigma^{i-1}) = \sigma\{\sigma^{i-1}(1)\ldots\sigma^{i-1}(m)\} \)

Since, here \( \sigma \) sends \( \sigma^{i-1}(i) \) to \( \sigma^i(1) \),

thus \( \sigma^i = (\sigma^{i-1}(1)\ldots\sigma^i(m)) \)

\( \implies \sigma^i = (\sigma^{i-1}(1)\ldots\sigma^i(m)) \)

Since \( \sigma^i(m) \equiv m+i \mod m \equiv i \mod m \) and \( \sigma^{i-1}(1) \equiv 1+i-1 \mod m \equiv i \mod m \)
\[ \sigma^i(m) = \sigma^{i-1}(1) \]
\[ \Rightarrow \sigma^i \text{ is an } m\text{-cycle.} \]

**Converse part**
Suppose \( \sigma^i \) is an \( m \)-cycle and suppose that \( (i, m) = d \geq 2 \). (we prove it by contradiction)
then there exists \( k,n \in \mathbb{N} \) such that \( i=kd \) and \( m=nd \),
since, \( (\sigma^i)^n = (\sigma^{kd})^n = \sigma^{kdn} = \sigma^{mek} = \sigma^m = I \)
where \( I \) is the identity permutation.
Hence \( |\sigma^i| \leq n < m \).
which is contradiction, since \( \sigma^i \) is an \( m \)-cycle and thus \( |\sigma^i| = m \). Thus \( i \) is relatively prime to \( m \).

5. **Que. No.05** Let \( n \geq 3 \). Prove the following in \( S_n \).

(a) Every permutation of \( S_n \) can be written as a product of at most \( n - 1 \) transpositions.

(b) Every permutation of \( S_n \) that is not a cycle can be written as a product of at most \( n - 2 \) transpositions.

**Proof (a) :**
We know that if \( k \geq 2 \), the cycle \((a_1, a_2, \ldots, a_k)\) can be written as \((a_1, a_k)(a_1, a_{k-1})\ldots(a_1, a_2)\)
which is \( k-1 \) transpositions.
Case-I, If \( k=1 \), then this cycle is the trivial cycle or the identity, which can be written as \( 1-1=0 \) transpositions.
Case-II, if \( k > 1 \),
we know that every permutation \( \sigma \in S_n \) can be written as a product of disjoint cycles, thus we can write
\[ \sigma = (a_{11}, a_{12}, \ldots, a_{1k_1})(a_{21}, a_{22}, \ldots, a_{2k_2})\ldots(a_{m1}, a_{m2}, \ldots, a_{mk_m}) \]
where \( k_1 + k_2 + \ldots + k_m = n \) and each of these cycle is disjoint.
we know that cycle \( i \) can be written as a product of \( k_i - 1 \) transpositions, and
\[ \sum_{i=1}^{m}(k_i-1) = \sum_{i=1}^{m} k_i - \sum_{i=1}^{m} 1 = n-m \]
this is maximized when \( m \) is minimized and the least value of \( m \) is 1.
Thus, the largest value of \( n-m \) can be \( n-1 \).

**Proof (b) :**
From part (a), \( \sigma = (a_{11}, a_{12}, \ldots, a_{1k_1})(a_{21}, a_{22}, \ldots, a_{2k_2})\ldots(a_{m1}, a_{m2}, \ldots, a_{mk_m}) \) where
\( \sum_{i=1}^{m} k_i = n \) and each of cycles is disjoint and also from (a), we still know that cycles \( i \) can be written as a product of \( k_i - 1 \) transpositions and
\[ \sum_{i=1}^{m}(k_i-1) = \sum_{i=1}^{m} k_i - \sum_{i=1}^{m} 1 = n-m \]
However, since \( \sigma \) is not a cycle, \( m \geq 2 \), thus \( n-m \) is maximized when \( m \) is minimized i.e \( m=2 \) i.e \( n-2 \) is the maximum value of \( n-m \).
Hence every permutation of \( S_n \) that is not a cycle can be written as a product of at most \( n-2 \) transpositions.
Que. No.06 Let $\sigma$ be a permutation of a set $A$. We say that $\sigma$ moves $a \in A$ if $\sigma(a) \neq a$. Let $S_A$ denote the permutations on $A$.

(a) If $A$ is a finite set then how many elements are moved by a $n$-cycle $\sigma \in S_A$?

(b) Let $A$ be an infinite set and let $H$ be the subset of $S_A$ consisting of all $\sigma \in S_A$ such that $\sigma$ only moves finitely many elements of $A$. Show that $H \leq S_A$.

(c) Let $A$ be an infinite set and let $K$ be the subset of $S_A$ consisting of all $\sigma \in S_A$ such that $\sigma$ moves at most 50 elements of $A$. Is $K \leq S_A$? Why?

**Proof (a):**
If $A$ is finite, then $\sigma$ moves only $n$ elements because $\sigma$ is $n$-cycle and the elements which is not in cycle are fixed.

**Proof (b):**
We may prove it by One-Step Subgroup Test.
As $A$ is infinite set and $\sigma \in S_A$ moves only finitely many elements of $A$. Since $H$ consists all $\sigma \in S_A$ 
$\Rightarrow$ $H$ is non-empty.
Now let, $\sigma \in H$ $\implies$ $\sigma^{-1} \in H$.
So, $\sigma \sigma^{-1} = I \Rightarrow H$.
Now checking for closure property,
Let $\sigma_1$ and $\sigma_2 \in H$ be any two permutations such that $\sigma_1$ and $\sigma_2$ both moves only finitely many elements of $A$.
Then $\sigma_1 \sigma_2$ also moves only finitely many elements of $A$.
$\Rightarrow$ Closure property holds.
$\Rightarrow H$ is subgroup of $A_5$.

**Proof (c):**
No, $K$ will not be subgroup of $S_A$.
Because, suppose that $\sigma_1$ moves at most 50 elements and $\sigma_2$ moves at most 50 elements, then $\sigma_1 \sigma_2$ (Product of two permutations) might moves more than 50 elements.
$\Rightarrow$ Closure property with respect to function composition is not satisfied in $K$.
$\Rightarrow K$ is not a subgroup of $S_A$.

Que. No.07 Show that if $\sigma$ is a cycle of odd length then $\sigma^2$ is a cycle.

**Proof:**  Suppose $\sigma : A \rightarrow A$ is a cycle with odd length. Then we can write $\sigma$ in a cycle notation as $\sigma$

$\sigma = (a_1, a_2, ..., a_{2k+1})$ where $a_1, a_2, ..., a_{2k+1} \in A$

On simple calculation, we may show that

$\sigma^2 = (a_1, a_2, ...a_{2k+1})(a_1, a_2, ...a_{2k+1})$

$\sigma^2 = (a_1, a_3, a_5, ...a_{2k+1}, a_2, a_4, ...a_{2k})$

$\Rightarrow \sigma^2$ is cycle whenever $\sigma$ is cycle.
8. **Que. No.08** Let $p$ be a prime. Show that an element has order $p$ in $S_n$ if and only if its cycle decomposition is a product of commuting $p$-cycles. Show by an explicit example that this need not be the case if $p$ is not prime.

**Proof :**

⇒ Suppose the order of $\sigma$ is $p(p$ is prime).  
Since order of $\sigma$ is the lcm of the sizes of the disjoint cycles in the cycle decomposition of $\sigma$, So all of these cycle must have sizes that divides $p$ is either 1 or $p$. 
Since 1-cycles are omitted from the notation for the cycle decomposition of $\sigma$. Thus the cycle decomposition consists entirely of $p$-cycles. Thus $\sigma$ is the product of disjoint commuting $p$-cycles.

⇐ Suppose $\sigma$ is the product of disjoint $p$-cycles. i.e $\sigma = c_1c_2c_3...c_r$ 
then $\sigma^p = (c_1c_2c_3...c_r)^2 = c_1^p c_2^p c_3^p...c_r^p = 1$ 
(since the $p^{th}$ power of $p$-cycles in $\sigma$ are all 1, so their product is 1) 
$\sigma^p = 1$ 
A $p$-cycle has order $p$, so no smaller power of $\sigma$ can be 1. Hence $|\sigma| = p$. 

For an example : 
Showing these conclusions may fail when $p$ is not a prime. 
Let $p=6$, $\sigma = (12)(345)$  
$|\sigma| = lcm(2, 3) = 6$  
but $\sigma$ is not the product of commuting 6-cycles.

9. **Que. No.09** Show that if $n \geq 4$ then the number of permutations in $S_n$ which are the product of two disjoint 2-cycles is $n(n-1)(n-2)(n-3)/8$.

**Solution :**

Given $n \geq 4$. 
Since, Permutations which are the product of two disjoint 2-cycles is of the form $(ab)(cd)$, i.e of length 4. 
Hence, there are $n$ choices for $a$, $(n-1)$ choices for $b$, $(n-2)$ choices for $c$ and $(n-3)$ choices for $d$. 
So there are $n(n-1)(n-2)(n-3)$ possible ways to write to write such a cycle. Since disjoint cycles commutes there, so there are 8 different ways that differently represent the same cycle.(As i mentioned it in sol. of Que.2) 
Hence total number of Permutation in $S_n$ which are the product of two disjoint 2-cyles is $\frac{(n)(n-1)(n-2)(n-3)}{8}$. 

10. **Que. No.10** Let $b \in S_7$ and suppose $b^4 = (2143567)$. Find $b$.

**Solution :**
\[ \therefore b \in S_7 \]
\[ |b| = 7 \]
\[ \Rightarrow b^7 = I \]

So \( b = Ib = (b^7).b = b^8 = (b^4)^2 \)
\[ \Rightarrow b = b^4.b^4 \]
\[ \Rightarrow b = (2143567)(2143567) = (2457136). \]

As given that \( b^4 = (2143567). \)

11. **Que. No. 11** Let \( b = (123)(145) \). Write \( b^{99} \) in disjoint cycle form.

   **Solution :**
   Since \( b = (123)(145) = (14523) \). So order of \( b \) is 5.
   (In case of single cycle, the order of permutation is the degree of permutation
   is the lengths of the set.)
   Now since \( |b| = 5 \), then \( b^5 = I \).
   So we can write \( b^{99} = (b^5)^{19}.b^4 = Ib^4 = b^4 = b^{-1}. \)
   Since \( b = (14523) \Rightarrow b^4 = b^{-1} = (32541) = (132541) \)
   so \( b^{99} = (13254) \) or \( (154)(132). \)

12. **Que. No. 12** Find three elements \( \sigma \) in \( S_9 \) with the property that \( \sigma^3 = (157)(283)(469). \)

   **Solution :**
   Let \( 1 = a_1, 2 = a_2, 3 = a_3, 4 = a_4, 5 = a_5, 6 = a_6, 7 = a_7 \) and 8 = \( a_8 \).
   Now we have to find \( \sigma \) such that \( \sigma^3 = (a_1a_5a_7)(a_2a_8a_3)(a_4a_6a_9) \)
   then \( \sigma_1 = (a_1 \ldots a_5 \ldots a_7 \ldots ) \)
   \( \sigma_1 = (a_1 a_2 \ldots a_5 a_8 \ldots a_7 a_3 \ldots ) \)
   \( \sigma_1 = (a_1 a_2 a_4 a_5 a_8 a_6 a_7 a_3 a_9) \)
   \( \sigma_1 = (1 2 4 5 8 6 7 3 9). \)
   Similarly we can find other two elements
   \( \sigma_2 = (a_1 \ldots a_5 \ldots a_7 \ldots ) \)
   \( \sigma_2 = (a_1 a_3 \ldots a_5 a_2 \ldots a_7 a_8 \ldots ) \)
   \( \sigma_2 = (a_1 a_3 a_9 a_5 a_2 a_4 a_7 a_8 a_6) \)
   \( \sigma_2 = (1 3 9 5 2 4 7 8 6). \)
   and
   \( \sigma_3 = (a_2 \ldots a_8 \ldots a_3 \ldots ) \)
   \( \sigma_3 = (a_2 a_1 a_4 a_8 a_5 a_6 a_3 a_7 a_9) \)
   \( \sigma_3 = (2 1 4 8 5 6 3 7 9). \)

13. **Que. No. 13** Show that if \( H \) is a subgroup of \( S_n \), then either every member of \( H \) is an even permutation or exactly half of the members are even.
Solution:
Let $H \subset S_n$ be any subgroup.
Now, we define $\overline{H} = \{ \sigma \in H \mid \sigma \text{ is even} \}$

Claim: $\overline{H}$ is subgroup of $H$.

Let $f,g \in \overline{H}$, Since $g$ are even, so $g^{-1}$ is also even.
Since the product of even permutations are still even, so we have $fog^{-1}$ is even.
So, here there are only two possibilities either $\overline{H} = H$ or $\overline{H} \subsetneq H$

Case-I, if $\overline{H} = H$, then we are done.

Case-II, if $\overline{H} \neq H$, then we need to show that $|\overline{H}| = \frac{|H|}{2}$

Since $\overline{H} \neq H$, it implies that there exists at least one odd permutation $\sigma \in H$

Now consider $f: \overline{H} \rightarrow \frac{H}{\overline{H}}$, defined by $f(h) = \sigma h$ for any $h \in \overline{H}$.

since $\sigma$ is odd and $h$ is even
⇒ $\sigma h$ is odd.
⇒ $\sigma h \in \frac{H}{\overline{H}}$

To prove that $\overline{H} = \frac{|H|}{2}$, We need to prove $f$ is 1-1 and onto.
for 1-1
let $h_1, h_2 \in H$ such that $h_1 = h2$.
since $h_1 = h_2$
⇒ $\sigma h_1 = \sigma h_2$ ⇒ $f(h_1) = f(h_2)$ ⇒ $f$ is 1-1.
and for onto
since $f^{-1}: \frac{H}{\overline{H}} \rightarrow \overline{H}$ is given by $f^{-1}(h) = \sigma^{-1}h'$ for every $h' \in \frac{H}{\overline{H}}$.

So $f$ is both 1-1 and onto
⇒ $|\overline{H}| = \frac{H}{|\overline{H}|}$, hence $|\overline{H}| = \frac{|H|}{2}$

14. Que. No.14 Suppose that $H$ is a subgroup of $S_n$ of odd order. Prove that $H$ is a subgroup of $A_n$. rate $S_n$.

Proof:
Let $H$ be a subgroup of $S_n$ of odd order.
i.e $|H| = \text{odd order}$
We may prove it by contradiction.
To the contrary, suppose $H \not\subseteq A_n$, then
suppose $\exists \sigma \in H$ such that $\sigma$ is an odd permutation.
Let $H = \{ \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_p \} \cup \{ \beta_1, \beta_2, \beta_3, \ldots, \beta_q \}$

$\therefore \sigma H = \{ \sigma \alpha_1, \sigma \alpha_2, \sigma \alpha_3, \ldots, \sigma \alpha_p \} \cup \{ \sigma \beta_1, \sigma \beta_2, \sigma \beta_3, \ldots, \sigma \beta_q \}$
15. Que. No.15 Prove that the smallest subgroup of $S_n$ containing (12) and (12...n) is $S_n$. In other words, these generate $S_n$.

Proof:

Let $\sigma = (12)$ and $\tau = (123...n)$

Suppose H is subgroup of $S_n$ which contains both $\sigma = (12)$ and $\tau = (123...n)$.

Now, we need to show that $H = S_n$.

Clearly, we have $H \subseteq S_n$. Since subgroups in particular are subsets.

Since we know that $S_n$ is generated by (n-1) transpositions $(12)(23)(34)(45)...(n-1 n)$.

Now, I want to show that (12) and (123...n) generates these (n-1) transposition.

Consider, $\tau \sigma \tau^{-1}$

$(12...n)(12)(12...n)^{-1} = (23)$

$(12...n)(23)(12...n)^{-1} = (34)$

$(12...n)(34)(12...n)^{-1} = (45)$

. . . . . . . . . . . . . .

$(12...n)(n-2 n-1)(12...n)^{-1} = (n-1 n)$

$(12...n)(n-1 n)(12...n)^{-1} = (n 1)$

Now i prove it by induction...

for n = 1, it is obviously true.

We assume that it is true for n = k, then

$(12...k)(k-1 k)(12...k)^{-1} = (k 1)$

Now, we wish to show that it is true for n = k+1

$(1, 2, ..., k, k+1)(k, k+1)(1, 2, ..., k, k+1)^{-1}$

$= (1, 2, ..., k, k+1)(k+1, k)(k+1, k, ..., 3, 2, 1)$

$= 6(1, 2, ..., k, k+1)(k+1, k, ..., 3, 2, 1)$

$= (k)(k-1)...(3)(2)(1, k+1)$

$= (k+1, 1)$

So, it is true for n=k+1

$\Rightarrow$ (12) and (123...n) generates $S_n$

Which shows that $S_n \subseteq H$.

Thus $h = S_n$

16. Que. No.16 Prove that for $n \geq 3$ the subgroup generated by the 3-cycles is $A_n$.

Proof:

Since every 3-cycle is an even permutation, then every 3-cycle of $S_n$ is in $A_n$. 

Now, Let $\tau \in A_n \Rightarrow \tau$ is an even permutation.

$\Rightarrow \tau$ is a product of an even no. of transposition.

However, $(a_1a_2)(a_3a_4) = (a_1a_2a_3)(a_2a_3a_4)$
And $(a_1a_2)(a_1a_3) = (a_1a_3a_2)$

Consequently, every product of two transposition (whether they share an element or not) can be written as a product of 3-cycles.

Hence, $\tau$ can be written as a product of 3-cycles.

$\Rightarrow$ For any $n \geq 3$, the subgroup generated by 3-cycle is $A_n$.

17. **Que. No.17** Prove that if a normal subgroup of $A_n$ contains even a single 3-cycle it must be all of $A_n$.

**Proof**:

Let $N \subseteq A_n$ be a normal subgroup and suppose that $(abc) \in N$. Let $\sigma' \in A_n$ be an arbitrary 3-cycles.

Then $\sigma' = \tau(abc)\tau^{-1}$ for some $\tau \in S_n$.

Now here, there are two possibility either $\tau \in A_n$ or $\tau \notin A_n$.

Case - I, If $\tau \in A_n$ then $\sigma' \in N$ and we are done.

Case - II, If $\tau \notin A_n$ then $\tau' = \tau(ab)$ is in $A_n$ and $\tau' = \tau(acb)\tau'^{-1}$ is once again in $N$.

$\Rightarrow$ If $N \leq A_n$ and contains a 3-cycle. Then $N = A_n$.

18. **Que. No.18** Prove that $A_5$ has no non-trivial proper normal subgroups. In other words show that $A_5$ is a simple group.

**Solution**:

Order of $A_5 = |A_5| = \frac{5!}{2} = 60 = 2^2.3.5$.

Let $N$ be proper normal subgroup of $A_5$, then $|N| = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30$.

Total no. of 5 order elements in $A_5 = \frac{5P5}{5} = 24$,

Total no. of elements of 3 order in $A_5 = \frac{5P3}{5} = 20$,

And total no. of 15-order elements in $A_5 = 0$.

Let us assume that $|H| = 3, 6, 12, 15$ then $|A_5| = 20, 10, 5, 4$

so $\gcd(3, |\frac{A_5}{H}|) = 1$

$\Rightarrow$ $H$ would contain all 20 elements of order 3.

Which is a contradiction.

{ As, Theorem says that If $H$ be normal subgroup of a finite group $G$. And if $\gcd \left( |x|, \frac{|G|}{|H|} \right) = 1$, then $x \in G$}. 

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Similarly, suppose that $|H| = 5, 10, 20$
then $\frac{A_5}{H} = 12, 6, 3$
$\implies H$ would contain all $24$ elements of order $5$.
which is a contradiction.
Let $|H| = 30$, then $\frac{A_5}{H} = 2$.
So again $\gcd\left(3, \frac{A_5}{H}\right) = 1$ and $\gcd\left(5, \frac{A_5}{H}\right) = 1$.
$\implies H$ would contain all $20 + 24 = 44$ elements.
we get again a contradiction.
And finally, let us assume that, $|H| = 2$ or $4$.
$\implies \frac{A_5}{H} = 30, 15$
Since, we know that any group of order $30$ or $15$ has an element of order $15$.
or As, if $\frac{A_5}{H} = 15 = 3 \times 5 = p \times q$ where $p=3$ and $q=5$.
( Theorem : If $G$ is a group of order $pq$, where $p$ and $q$ are primes, $p < q$ and $p \nmid q$, then $G$ is cyclic.)
$\Rightarrow G$ has at least one element of order $15$.
Which is again contradiction,
because $A_5$ contains no such element, neither does $\frac{A_5}{H}$.
This proves that $A_5$ is simple.

19. Que. No.19 Show that $Z(S_n)$ is trivial for $n \geq 3$.

**Solution :**
Let $\sigma \in S_n$ be a non-identity element then there exists two distinct $a,b \in \{1,2,3,...,n\}$ with $\sigma(a) = b$.
Since $n \geq 3$, Now choosing $k \in \{1,2,3,...,n\}$ such that $k \neq a$ and $k \neq b$.
Let $\tau = (ak)$. Then
$\tau(\sigma(a)) = \tau(b) = k$ and $\sigma(\tau(a)) = \sigma(a) = b$
since $k \neq b \Rightarrow \tau(\sigma(a)) \neq \sigma(\tau(a))$.
Hence for every non-identity permutation in $S_n$, there exists some element not commuting with it.
Therefore $Z(S_n)$ must be trivial.

20. Que. No. 20 Show that two permutations in $S_n$ are conjugate if and only if they have the same cycle structure or decomposition. Given the permutation $x = (12)(34), y = (56)(13)$, find a permutation $a$ such that $a^{-1}xa = y$.

**Proof :**
For any $\sigma$ and any $d \leq n$, we have
$\sigma(12...d)\sigma^{-1} = (\sigma(1)\sigma(2)....\sigma(d))$
This shows that any conjugate of d-cycle is again d-cycle. Since every permutation is a product of disjoint cycles, it follows that the cycle structure of conjugate permutations are the same. In other direction, let \( \tau = (a_1 a_2 \ldots a_r) (a_{r+1} a_{r+2} \ldots a_s) \ldots (a_l \ldots a_m) \) and \( \tau' = (a'_1 a'_2 \ldots a_r) (a_{r+1} a_{r+2} \ldots a_s) \ldots (a_l \ldots a_m) \) be two permutations having the same cycle structure. Define \( \sigma \in S_n \) by \( \sigma(a'_i) = a'_i \) for \( i = 1, 2, \ldots, m \) then \( \sigma \tau \sigma^{-1} = \sigma(a_1 a_2 \ldots a_r) \sigma^{-1} \sigma(a_{r+1} a_{r+2} \ldots a_s) \sigma^{-1} \ldots \sigma(a_l \ldots a_m) \sigma^{-1} = (a'_1 a'_2 \ldots a_r) (a_{r+1} a_{r+2} \ldots a_s) \ldots (a_l \ldots a_m) = \tau' \). This shows that \( \tau \) and \( \tau' \) are conjugate. Now, given the permutation \( x = (12)(34), y = (56)(13) \) since that \( a^{-1}xa = y \). \( \therefore xa = ay \Rightarrow x = aya^{-1} \). \( \Rightarrow ((12)(34)) = a((56)(13))a^{-1} \). \( \Rightarrow ((12)(34))(5)(6) = a((56)(13)(2)(4))a^{-1} \). \( \Rightarrow 1 = a(5), 2 = a(6), 3 = a(1), 4 = a(3) \) and \( 5 = a(2), 6 = a(4) \). \( \Rightarrow a = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 6 & 1 & 2 \end{array} \). \( \Rightarrow a = (134625) \). Checking for \( a, a = (134625) \) and \( a^{-1} = (526431) = (152643) \) \( \therefore a^{-1}xa = (134625)((12)(34))(152643) \). \( = (13)(2)(4)(56) = (13)(56) = \text{RHS, Hence done.} \)

**References**


